EXTENSIVE MEASUREMENT WHEN CONCATENATION IS RESTRICTED AND MAXIMAL ELEMENTS MAY EXIST[1]

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1. INTRODUCTION

Theories of fundamental measurement begin with purported qualitative laws about observable relations among certain entities, and using these laws a numerical representation is constructed which reflects their formal structure. The best known example, extensive measurement, arose from an analysis of such familiar physical measures as mass and length. In measuring mass, the classical theory supposes that we can ascertain which of two entities has the greater mass by comparing them, for example, on an equal-arm pan balance in a vacuum—the one in the pan that drops having, by definition, the greater mass. If the comparison of x with y shows that x has more than or an amount equivalent to y of the attribute being measured, we write xRy. Furthermore, the theory assumes that new entities can be generated from old ones by placing two or more of the old ones on a single pan. The latter combining operation is called "concatenation," and the new entity generated by concatenating x with y is denoted by xoy.

Various plausible and, to some degree, empirically true assumptions about o, R, and their interconnections are stated (for a summary of them, see Definition 2 and the discussion following it), from which it is shown that we can assign to each entity x a real number $\varphi(x)$ in such a way that the function φ has the following three properties. First, it is order preserving (monotonic),

$$xRy \text{ if and only if } \varphi(x) \geqq \varphi(y);$$

second, it is additive over concatenations,

$$\varphi(xoy) = \varphi(x) + \varphi(y);$$

and third, any other assignment having the first two properties differs from φ only by a positive multiplicative factor, i.e., the family of representations forms a ratio scale.

Systematic discussions of these ideas have been given by, among others, Campbell (1920, 1928), Nagel (1932), Suppes (1951), and Suppes and Zinnes (1963).

Among the various idealizations embodied in the traditional theory of extensive measurement, one that is rarely, if ever, experimentally fulfilled, even approximately, is the postulated freedom to concatenate any two elements of the system to form a third that is also within the system. A simple finite induction shows that if x is in the system and n is a positive integer, then concatenations of n elements each of which is equivalent to x is also in the system. Denoting this element by nx, then if an additive representation φ exists, we have $\varphi(nx) = n\varphi(x)$. Thus, in theory, φ is unbounded. In measuring mass with a pan balance, one obviously cannot concatenate freely without either damaging the balance or running out of space.

It appears that most theorists believe that this practical limitation places an unavoidable limit on the precision of measurement: specifically, that if this limitation were incorporated into the theory it would either make it impossible to construct a numerical representation or, if the construction were possible, two representations would not necessarily be related by a similarity transformation, i.e., the theory would not lead to a ratio scale representation. As we shall see, neither alternative is true. Precise ratio scales over finite sets of elements are possible provided that the usual axioms are modified slightly. The basic fact is that it is unnecessary to construct actual concatenations of indefinitely many replicas of a given element in order to achieve precision; five subdivisions do just as well, as has been accepted in practice.

The single major exception to the comments of the last paragraph is Behrend (1956), whose work was only brought to our attention (by Richard Robinson, whom we thank) after the present paper was completed. Behrend gave a system much like that of Definition 2, and he proved the same representation as given in Theorem 5. We cite the differences between the two systems following Definition 2. Behrend's proof is substantially the same as ours. Nonetheless, we prove the result here for three reasons: first, for completeness; second, because Behrend's paper does not seem to be widely known; and third, because our proof differs in some ways, including the fact that it covers the case where indifference is not equality. (A similar development for additive conjoint measurement [Luce and Tukey, 1964] can be found in Luce, 1966.)

A second limitation of the traditional theory is its failure, even when there is no restriction on concatenation, to take into account the possibility that the system may have a maximal element. The best known example is velocity: according to the theory of relativity, no velocity may exceed that of light. This is true in spite of the fact that, in principle at least, any two velocities may be concatenated (superimposed) to form a new one; it simply means that they summate in a particular way so that the resultant velocity remains less than or equal to that of light. With this example in mind, we also wish to modify the axioms so as to admit the possibility that maximal elements may exist.

2. THE AXIOMS

The axiom system given in Definition 2 is a modification of Suppes' (1951) and Behrend's (1956) systems; the exact nature of the modification is pointed out following the formal statement of the system.

Definition 1. *Let* A *be a non-empty set,* B *a non-empty subset of* $A \times A$, R *a binary relation on* A, *and* o *a binary function from* B *into* A. *An element* $a \varepsilon A$ *is* maximal relative to R and o *if, for all* $x \varepsilon A$, aRx, *and there is some* $x \varepsilon A$ *such that* $(a,x) \varepsilon B$.

For brevity, we sometimes refer to an element simply as maximal without specifying that it is relative to R and o.

Definition 2. *Let* A *be a non-empty set,* B *a non-empty subset of* $A \times A$, R *a binary relation on* A, *and* o *a binary function from* B *into* A. *The quadruple* $<A,B,R,o>$ *is called an* extensive system *if, for all* $x,y,z \varepsilon A$, *the following five axioms hold:*

1. R *weakly orders* A.
2. *If* $(x,y) \varepsilon B$ *and* $(xoy,z) \varepsilon B$, *then* $(y,z) \varepsilon B$, $(x,yoz) \varepsilon B$, *and* (xoy)ozRxo-(yoz).
3. *If* $(x,z) \varepsilon B$ *and* xRy, *then* $(z,y) \varepsilon B$ *and* xozRzoy.
4. *If not* xRy, *then there exists an element* $y\text{-}x \varepsilon A$ *such that* $(x,y\text{-}x) \varepsilon B$, yRxo(y-x), *and* xo(y-x)Ry.
5. *Let* n *be a positive integer. For* $n=1$, *define* $1x=x$. *For* $n > 1$, *if* $(n\text{-}1)x$ *is defined and* $((n\text{-}1)x,x) \varepsilon B$, *then define* $nx = (n\text{-}1)xox$. *For all non-maximal* $y \varepsilon A$ *and all* $x \varepsilon A$, *the set*
$$\{n \mid n \text{ is a positive integer and } yRnx\}$$
is finite.

The major changes introduced into Suppes' system are these:

(i) R is assumed to be a weak order, not just transitive; Suppes' proof that R is connected depends on properties of o that we no longer possess, and so we are forced to add that property explicitly.

(ii) Suppes' Axiom 2 says, in essence, that $B = A \times A$. We have weakened it considerably by requiring the properties specified in Axioms 2-4. Among other things, it is shown in Lemma 1 that if x and y can be concatenated, then so can y and x and so can any pair of elements that, under R, are dominated by x and y. Both of these conditions are likely to be satisfied by any empirically interesting interpretation of the system.

(iii) Axioms 2–4 are the same as Suppes' Axioms 3–5 provided only that the relevant concatenations are possible.

(iv) Suppes' system includes between our Axioms 4 and 5 the axiom that for all $x,y \varepsilon A$, not xRxoy. As we show in Theorem 1, the somewhat weaker statement xoyRx follows from the other axioms, and in Theorem 2 we show that Suppes' axiom is equivalent to the assumption that there are no maximal elements.

(v) With unrestricted concatenation, the Archimedean Axiom 5 can be, and usually is, stated as follows: if yRx then there exists a positive integer n such that nxRy. In the presence of the other axioms and with unrestricted concatenations (i.e. $B=A\times A$), it is clear that our formulation is equivilent to the usual one. When concatenation is restricted, the usual formulation is meaningless and so something like our Axiom 5 is needed. Using velocity as a guide, we have assumed that the Archimedean property holds only for non-maximal y. Of course, if B is finite, as it will be in practice, Axiom 5 is satisfied trivially.

The relation to Behrend's (1956) system is as follows.

(i) Let xIy if xRy and yRx and xPy if xRy and not yRx. Behrend assumed I to be equality, and he used two of its usual properties, specifically, substitutability and the fact that it is an equivalence relation. For this reason, he needed only to assume that R is connected; however, with I different from equality we must assume R is a weak ordering. In particular, his proof that P is transitive is no longer valid.

(ii) Behrend stated Axiom 2 in terms of I with the existence of the concatentations on the right implying those on the left, rather than the other way round.

(iii) Instead of Axiom 3, Behrend assumed: if $(x,z)\varepsilon B$, then xIy if and only if $(y,z)\varepsilon B$ and xozIyoz; and if $(z,x)\varepsilon B$, then xIy if and only if $(z,y)\varepsilon B$ and zoxIzoy. He showed that the analogous statements hold for P and that I is commutative.

(iv) Axiom 4 is unchanged.

(v) As in Suppes' system, the property xoyPx is assumed.

(vi) His Archimedean condition is essentially as we have given it.

The following simple example establishes that these changes are significant in the sense that a system with very few elements can fulfill the axioms, an additive representation exists, and it is a ratio scale:

$$\begin{aligned}
&A = \{a,b,c,d,e\},\\
&B = \{(e,e),\ (d,e),\ (e,d),\ (c,e),\ (e,c)\},\\
&o{:}\,eoe = c,\ doe = eod = b,\ coe = eoc = a,\\
&R{:}\,aIbPcIdPe,
\end{aligned}$$

where P and I have their usual meanings and R includes all implications that follow from transitivity. It is routine to see that the axioms are fulfilled, that if φ satisfies

$$\varphi(a) = \varphi(b) = 3\varphi(e),\ \varphi(c) = \varphi(d) = 2\varphi(e),$$

where $\varphi(e) > o$, it is an extensive (additive over o) representation, and that any extensive representation is of this form.

3. A PROPERTY OF MAXIMAL ELEMENTS

Throughout this section the axioms of Defintion 2 are assumed to hold, and P and I are defined in terms of R in the usual way.

Lemma 1. *If* (x,y)εB, *then* (y,x)εB; *and if in addition* xRu *and* yRV, *then* (u,v)εB *and* xoyRuov.

Proof. By Axiom 1, xRx and so by Axiom 3, (y,x)εB. Since xRu, Axiom 3 yields (y,u)εB and xoyRyou. Similarly, (u,v)εB and youRuov, and so by Axiom 1, xoyRuov. QED

Lemma 2. *If* (x,y)εB, *then* xoyIyox.

Proof. By Lemma 1, (y,x)εB. Since, by Axiom 1, xRx, Axiom 3 yields xoyRyox. Similarly, yRy yields yoxRxoy. QED

Lemma 3. *If* (x,y), (xoy,z)εB, *then* (xoy)ozIxo(yoz).

Proof. Immediate from Axiom 2 and Lemma 2. QED

Note that Lemmas 2 and 3 imply that the ordering and grouping of concatenations is a mere matter of convenience, once we have assured their existence.

Theorem 1. *If* x,yεA *and* (x,y)εB, *then* xoyRx.

Proof. Suppose that, contrary to the assertion, xPxoy. By Axiom 4 and Lemma 3, there exists z = (x-xoy)εA such that xI(xoy)ozIxo(yoz). By Lemma 1 and Axiom 3, xPxoyR(xoy)oyIxo2y, and so by a finite induction, xPxony for all positive integers n. Putting these two observations together, xo(yoz)IxPxony, and so by Axioms 1 and 3, yozRny. Axiom 5 implies, therefore, that yoz is maximal, and in particular yozRx. Thus, by Lemma 1 and Axiom 3, xIxo(yoz)Rxox, and so by another finite induction and Axiom 5 we conclude that x is also maximal. Thus, xRy. Suppose that xPy, then since xR(x-y), Axioms 3 and 4 imply that yo(x-y) IxPxoyR(x-y)oy, which is impossible by Lemma 2 and Axiom 1. So yIx, and thus xPxox. Since x is maximal and xPxoy, Axioms 3 and 4 yield xoxR(xoy)o(x-xoy)Ix, which contradicts xPxox. QED

Lemma 4. *If* a,bεA *are both maximal, then* aIb.

Proof. Trivial. QED

Theorem 2. *If* a,xεA *and* (a,x)εB, *then* a *is maximal relative to* R *and* o *if and only if* aIaox.

Proof. If a is maximal, then by Defintion 1, aRaox. By Theorem 1, aoxRa; hence, aIaox.

Conversely, suppose that aIaox and that a is not maximal. Since (a,x)εB, Lemma 1 implies that (aox,x)εB and so by Axiom 3 and Lemma 3, aIaoxI(aox) oxIao2x. By induction, for every positive integer n, nx exists and aIaonx. Since, by assumption, a is not maximal, Axiom 5 implies that, for some n, nxPa. So, by Axiom 3, aIaonxRaoa. Coupled with Theorem 1, this means aIaoa. A finite induction then shows that for every positive integer m, ma exists and aIma. As this is impossible by Axiom 5, it follows that a must be maximal. QED

Corollary. *In an extensive system, the assumption that there is no maximal element relative to* R *and* o *is equivalent to the assertion that for all* x,yεA *such that* (x,y)εB, *then not* xRxoy.

Proof. Theorems 1 and 2. QED

4. PRELIMINARY RESULTS WHEN NO MAXIMAL ELEMENT EXISTS

Throughout this and the next section we assume the first four Axioms of Definition 2 and the property (stated in the Corollary to Theorem 2) that not xRxoy. Axiom 5 is not used again until Section 6, and then only once.

Lemma 5. *If* (x,u), (y,u)εB *and* xouRyou, *then* xRy.

Proof. Suppose, on the contrary, not xRy, then by Axiom 4, yIxo(y-x). Thus, xouRyouIxo(y-x)ouI(xou)o(y-x), which contradicts the Corollary to Theorem 2. QED

Lemma 6. *If* (x,y)εB, xPu, *and* yPv, *then* (x-u,y-v)εB *and* (x-u)o(y-v) I(xoy-uov).

Proof. By the definitions of x-u and y-v and Theorem 1, xR(x-u) and yR(y-v), and so by Lemma 1 (x-u,y-v)εB. By Lemmas 1, 2, and 3,

xoyIuo(x-u)oyI(x-u)o(uoy),
uoyIuovo(y-v)I(y-v)o(uov).

So, by Lemma 1,

xoyI(x-u)o(y-v)o(uov).

But, by definition,

xoyI(xoy-uov)o(uov),

and the result follows by Axiom 1 and Lemma 5. QED

Corollary. *If* (x,y)εB *and* xPu, *then* (x-u)I(xoy-uoy).

Proof. From the proof of Lemma 6,

(x-u)o(uoy)IxoyI(xoy-uoy)o(uoy),

and the result follows by Lemma 5. QED

Lemma 7. *If* (mx,nx)εB, *then* (m+n)x *is defined and* (m+n)xImxonx.

Proof. By induction and using Lemmas 2 and 3,

mxonxImxo [(n-1)xox]I(m+1)xo(n-1)x . . . . I(m+n)x. QED

Lemma 8. *There exists* eεA *such that* (e,e)εB.

Proof. Since B is non-empty, there exists (x,y)εB. Either xRy or yRx.

If the former, then it with yRy implies, by Lemma 1, that $(y,y)\varepsilon B$. If the latter, $(x,x)\varepsilon B$. QED

Corollary. *If* eRx *and* eRy, *then* $(x,y)\varepsilon B$.

Lemma 9. *If* eRx, eRy, eRz, xoyPe, *and* yoxPe, *then* xo(yoz-e)I-(xoy-e)oz.

Proof. Since eoeRxoyI(xoy-e)oe, Lemma 5 yields eR(xoy-e). Similarly, eR(yoz-e). Hence by the Corollary to Lemma 8, the asserted concatenations exist. Let u=e-y. If uRz, then by Lemma 1 we obtain eIuoyRzoy, contrary to assumption; so zPu. By the definition of z-u, xozIxo(z-u)ou, and so by the definition of xoz-u, (xoz-u)Ixo(z-u). From the Corollary to Lemma 6, (z-u)I(yoz-uoy)I(yoz-e); hence by Lemma 1, (xoz-u) Ixo(yoz-e). In a similar manner, (xoz-u)I(xoy-e)oz, and the result follows by Axiom 1. QED

Lemma 10. *If* eRx, eRy, eRz, xoyPe, *and* eRyoz, *then* $(x,yoz)\varepsilon B$, xoyozPe, *and* (xoy-e)ozI(xoyoz-e).

Proof. From eRyoz and eRx, the Corollary to Lemma 8 shows that $(x,yoz)\varepsilon B$. Moreover, if eRxoyoz, then Theorem 1 yields eRxoy, contrary to assumption. Finally

$$
\begin{aligned}
(xoy\text{-}e)ozoeI(xoy\text{-}e)oeoz & \\
Ixoyoz & \\
I(xoyoz\text{-}e)oe, &
\end{aligned}
$$

and the result follows by Lemma 5. QED

5. CONSTRUCTION OF AN ORDINARY EXTENSIVE SYSTEM

Definition 3. *Suppose that* $a = \langle A,B,R,o \rangle$ *is an extensive system and let* $e\varepsilon A$ *be a fixed element for which* $(e,e)\varepsilon B$ *(see Lemma* 8*). The system* $a_e = \langle A_e,R_e,* \rangle$ *is defined by*:

$A_e = \{(m,x) | m$ is a non-negative integer, $x\varepsilon A$, and eRx$\}$.

R_e: $(m,x)R_e(n,y)$ if either $m > n$ or $m = n$ and xRy.

$$
:\ (m,x)(n,y) = \begin{cases} (m+n, xoy) & \text{if eRxoy} \\ (m+n+1, xoy\text{-}e) & \text{if xoyPe.} \end{cases}
$$

Note that $*$ is well defined since, by the Corollary to Lemma 8, eRx and eRy imply $(x,y)\ \varepsilon B$; and if xoyPe then eR(xoy-e) since the converse, (xoy-e)Pe, leads to the contradiction

$$xoyI(xoy\text{-}e)oePeoeRxoy.$$

It is clear that $*$ is commutative.

Theorem 3. *If Axioms* 1–4 *hold and if there is no element that is maximal relative to* R *and* o, *then the system* a_e *satisfies Suppes' axioms for an extensive system.*

Proof. 1. R_e is obviously a weak order.

2. $*$ is obviously a function from $A_e \times A_e$ into A_e.

3. To show the associativity of $*$, observe that by definition,

$$[(m,x)*(n,y)]*(p,z) = \begin{cases} [m+n+p, x\circ y\circ z] & \text{if } eRx\circ y \text{ and } eRx\circ y\circ z \\ [m+n+p+1, (x\circ y\text{-}e)\circ z] & \text{if } x\circ yPe \text{ and } eR(x\circ y\text{-}e)\circ z \\ [m+n+p+1, x\circ y\circ z\text{-}e] & \text{if } eRx\circ y \text{ and } x\circ y\circ zPe \\ [m+n+p+2, (x\circ y\text{-}e)\circ z\text{-}e] & \text{if } x\circ yPe \text{ and } (x\circ y\text{-}e)\circ zPe \end{cases}$$

and

$$(m,x)*[(n,y)*(p,z)] = \begin{cases} [m+n+p, x\circ y\circ z] & \text{if } eRy\circ z \text{ and } eRx\circ y\circ z \\ [m+n+p+1, x\circ(y\circ z\text{-}e)] & \text{if } y\circ zPe \text{ and } eRx\circ(y\circ z\text{-}e) \\ [m+n+p+1, x\circ y\circ z\text{-}e] & \text{if } eRy\circ z \text{ and } x\circ y\circ zPe \\ [m+n+p+2, x\circ(y\circ z\text{-}e)\text{-}e] & \text{if } y\circ zPe \text{ and } x\circ(y\circ z\text{-}e)Pe \end{cases}$$

There are four cases:

(i) If $eRx\circ y$ and $eRx\circ y\circ z$, then, by the Corollary to Theorem 2, $eRy\circ z$, and so the first line holds in each case and they are identical.

(ii) If $x\circ yPe$ and $eR(x\circ y\text{-}e)\circ z$, then either $y\circ zPe$ or $eRy\circ z$. If the former, $eR(x\circ y\text{-}e)\circ z$ together with Lemma 9 yields $eRx\circ(y\circ z\text{-}e)$, and so the second line holds in each case and they are equivalent. If the latter, Lemma 10 yields $x\circ y\circ zPe$. So the second line of the first expression and the third line of the second expression hold and, using Lemma 10, these expressions are equivalent.

(iii) If $eRx\circ y$ and $x\circ y\circ zPe$, then either $eRy\circ z$ or $y\circ zPe$. If the former, the third line holds in each case and they are identical. If the latter, we show $eRx\circ(y\circ z\text{-}e)$ in which case the second line of the second expression holds and, using Lemma 10, it is equivalent to the third line of the first expression. Suppose that $x\circ(y\circ z\text{-}e)Pe$, then $x\circ y\circ zIx\circ(y\circ z\text{-}e)\circ ePe\circ eR(x\circ y)\circ z$, a contradiction.

(iv) If $x\circ yPe$ and $(x\circ y\text{-}e)\circ zPe$, then $y\circ zPe$. For suppose, on the contrary, that $eRy\circ z$, then with eRx we obtain the contradiction $e\circ eRx\circ y\circ zI(x\circ y\text{-}e)\circ e\circ zPe\circ e$. Lemma 9 then yields $x\circ(y\circ z\text{-}e)I(x\circ y\text{-}e)\circ zPe$, and so the fourth line of the second expression also holds and, using Lemma 9, these expressions are equivalent.

4. Suppose that $(m,x)R_e(n,y)$, then we wish to show that $(m,x)*(p,z)R_e(n,y)*(p,z)$. If $m > n$, then there are four cases:

(i) If $eRx\circ z$ and $eRy\circ z$, then $(m,x)*(p,z) = (m+p, x\circ z)R_e(n+1+p, x\circ z)R_e(n+p, y\circ z) = (n,y)*(p,z)$.

(ii) If $eRx\circ z$ and $y\circ zPe$, then $x\circ zR(y\circ z\text{-}e)$ since otherwise we obtain $y\circ zP(y\circ z\text{-}e)\circ eRx\circ z\circ e$, whence, by Lemma 5, $yPx\circ ePe$, contrary to choice of y. Thus, $(m,x)*(p,z) = (m+p, x\circ z)R_e(n+1+p, x\circ z)R_e(n+1+p, y\circ z\text{-}e) = (n,y)*(p,z)$.

(iii) If $x\circ zPe$ and $eRy\circ z$, then $(m,x)*(p,z) = (m+p+1, x\circ z\text{-}e)R_e(n+p+2, x\circ z\text{-}e)R_e(n+p, y\circ z) = (n,y)*(p,z)$.

(iv) If xozPe and yozPe, then
$(m,x)*(p,z) = (m+p+1,yoz\text{-}e)R_e(n+p+2,xoz\text{-}e)R_e(n+p+1,yoz\text{-}e) = (n,y)*(p,z)$.

Alternatively, m = n and xRy, in which case there are again four cases:

(i) If eRxoz and eRyoz, then the result is immediate by Axiom 3.

(ii) The case where eRxoz and yozPe is impossible since Axiom 3 and xRy imply the contradiction eRxozRyozPe.

(iii) If xozPe and eRyoz, then
$(m,x)*(p,z) = (m+p+1,xoz\text{-}e)I_e(n+p+1,xoz\text{-}e)R_e(n+p,yoz) = (n,y)*(p,z)$.

(iv) If xozPe and yozPe, then
$(m,x)*(p,z) = (m+p+1,xoz\text{-}e)I_e(n+p+1,xoz\text{-}e)R_e(n+p+1,yoz\text{-}e) = (n,y)*(p,z)$ because the supposition (yoz-e)P (xoz-e) yields

$$yozI(yoz\text{-}e)oeP(xoz\text{-}e)oeIxoz,$$

which, by Lemma 5, implies yPx, contrary to assumption.

5. Suppose that not $(m,x)R_e(n,y)$, i.e., $(n,y)P_e(m,x)$, then we show that some (p,z) exists for which $(n,y)I_e(m,x)*(p,z)$. If $n > m$, then either yPx, in which case $(m,x)*(n\text{-}m,y\text{-}x)I_e(n,y)$; or xIy, in which case $(m,x)*(n\text{-}m\text{-}1,e)I_e(m,xoe\text{-}e)I_e(n,y)$; or xPy, in which case $(m,x)*(n\text{-}m\text{-}1, eoy\text{-}x)I_e(n,eoy\text{-}e)I_e(n,y)$. If $n = m$ and yPx, then $(m,x)*(0,y\text{-}x)I_e(n,y)$.

6. Next we show that not $(m,x)R_e(m,x)*(n,y)$.
By definition,

$$(m,x)*(n,y) = \begin{cases}(m+n,xoy) \text{ if } eRxoy\\ (m+n+1,xoy\text{-}e) \text{ if } xoyPe.\end{cases}$$

If xoyPe, then $m+n+1 > m$ and we are done. If eRxoy, the same argument holds if $n > 0$. If $n = 0$, then $(m,x)*(0,y)P_e(m,x)$ because xoyPx by the Corollary to Theorem 2.

7. Suppose that $(m,x)R_e(n,y)$. Choose any integer k such that $kn > m$, then

$$(k+1)(n,y)R_e(kn,y)R_e(m,x). \qquad \text{QED}$$

6. IMBEDDING OF $\mathcal{a}$/I IN A_e/I_e

Throughout this section we deal only with the systems that result by treating as elements the equivalence classes under, respectively, the equivalence relations I and I_e. Letters from the begining of the alphabet will be used to denote these classes and = replaces I and I_e; e denotes both the element of Definition 3 and its equivalence class. We now invoke all five axioms of Definition 2 plus the assumption of no maximal elements.

Definition 4. The subsystem $\mathfrak{a}'_e = \langle A'_e, B'_e, R'_e, *' \rangle$ of a_e *is defined by*:

$$
\begin{aligned}
A'_e &= \{(N_a, a_e) \mid a \varepsilon A/I,\ N_a = \max\{n \mid aPne\},\ a_e = a\text{-}N_a e\},\\
B'_e &= \{((N_a, a_e),\ (N_b, b_e)) \mid (a,b) \varepsilon B/I\},\\
R'_e &\quad \text{is the restriction of } R_e \text{ to } A'_e,\\
*' &\quad \text{is the restriction of } * \text{ to } B'_e.
\end{aligned}
$$

Note that by Axiom 5, N_a exists.

Lemma 11. eRa_e.

Proof. Suppose that a_ePe. Since $(a_e, N_ae) \epsilon B/I$, it follows from Lemma 1 that $(e, N_ae) \epsilon B/I$, and so

$$a = N_aeoa_ePN_aeoe = (N_a + 1)e,$$

contrary to the choice of N_a. QED

Theorem 4. *If* a *is an extensive system with no element that is maximal relative to* R *and* o, *then the subsystem* $\mathfrak{a}'_e$ *is isomorphic to* a/I.

Proof. The mapping $a \longleftrightarrow (N_a, a_e)$ is 1:1. For suppose $N_a = N_b = N$ and $a_e = b_e$, then a-Ne = b-Ne and so a = (a-Ne)oNe = (b-Ne)oNe = b. The converse is obvious.

The mapping is order preserving. If aRb, then clearly $N_a \geqq N_b$. If $N_a > N_b$, then $(N_a, a_e)R'_e(N_b, b_e)$. If $N_a = N_b = N$, then $a_e = (a\text{-}Ne)R(b\text{-}Ne) = b_e$, and again the result follows. The converse is similar.

The mapping preserves o. Suppose that $(a,b) \epsilon B/I$ and c = aob. Since aRN_ae and bRN_be, then by Lemma 7, $aobRN_aeoN_be = (N_a + N_b)e$. Thus, $N_c \geqq N_a + N_b$. Let $N_c - N_a + N_b = k$. By Lemma 6,

$$a_eob_e = [aob\text{-}(N_a + N_b)e]P[N_ce\text{-}(N_a + N_b)e] = ke.$$

So, if eRa_eob_e, $k = 0$, i.e., $N_c = N_a + N_b$ and $a_eob_e = c_e$. Otherwise, a_eob_ePe. Since $eoeRa_eob_e$, it follows that $k \leqq 1$. But since $[aob\text{-}(N_a + N_b)e]Pe$, $aobP(N_a + N_b + 1)e$, and so $k = 1$, from which the result follows.

Conversely, suppose that

$$(N_c, c_e) = (N_a, a_e)*(N_b, b_e) = \begin{cases} (N_a + N_b, a_eob_e) \text{ if } eRa_eob_e \\ (N_a + N_b + 1, a_eob_e\text{-}e) \text{ if } a_eob_ePe. \end{cases}$$

In the first case,

$$
\begin{aligned}
c &= c_eoN_ce\\
&= a_eob_eo(N_a + N_b)e,
\end{aligned}
$$

and in the second

$$
\begin{aligned}
c &= c_eoN_ce\\
&= (a_eob_e\text{-}c)o(N_a + N_b + 1)e\\
&= (a_eob_e)o(N_a + N_b)e.
\end{aligned}
$$

But, by Lemma 6,

$$\begin{aligned}(a_e o b_e) o (N_a + N_b) e &= (a\text{-}N_a e) o (b\text{-}N_a e) o (N_a + N_b) e \\ &= [aob\text{-}(N_a + N_b) e] o (N_a + N_b) e \\ &= aob. \end{aligned}$$

QED

7. REPRESENTATION AND UNIQUENESS THEOREMS

Theorem 5. *If $\mathfrak{a} = \langle A,B,R,o \rangle$ is an extensive system with no element that is maximal relative to* R *and* o, *then there exists a positive real-valued function* φ *on* A *such that*

(i) xRy *if and only if* $\varphi(x) \geqq \varphi(y)$;
(ii) *if* $(x,y)\epsilon B$, $\varphi(xoy) = \varphi(x) + \varphi(y)$.

Proof. Suppes (1951) proved the existence of such a function for any system fulfilling his axioms, in particular for $\mathfrak{a}_e$. The restriction of it to $\mathfrak{a}'_e$ transformed isomorphically by Theorem 4 to $\mathfrak{a}$ completes the proof. QED

Theorem 6. *If φ and ψ are two functions fulfilling Theorem* 5, *then there exists a constant $a >$ o such that $\varphi = \alpha\psi$.*

Proof. With no loss of generality, we may suppose that $\varphi(e) = \psi(e) = 1$ and then show $\varphi = \psi$. Suppose that $\varphi(x) \neq \psi(x)$, then since $x_e o N_x e I x$, we see immediately that $\varphi(x_e) \neq \psi(x_e)$. But since in $\mathfrak{a}_e$, $\varphi(n,x) = n + \varphi(x)$, this implies non-unique additive scales on $\mathfrak{a}_e$, contrary to Suppes' uniqueness theorem. QED

Theorem 7. *If $\langle A,B,R,o \rangle$ is an extensive system with no element that is maximal relative to* R *and* o, *and if* B *is finite, then one, and so an infinity, of the representations of Theorem* 5 *is into the positive integers.*

Proof. Since B is finite, so is the set

$$C = \{x | x \epsilon A \text{ and there exists } y \epsilon A \text{ such that } (x,y) \epsilon B\}.$$

Choose e to be the least element of C under the ordering R. By Lemma 1, $(e,e)\epsilon B$; moreover, if $a \epsilon A$, then aRe since if ePa then, by Lemma 1, $(a,a)\epsilon B$, which contradicts the choice of e. Thus, by Lemma 11 the elements of A'_e of Definition 4 must each be of the form (N_x,e), where $x \epsilon A/I$ and N_x is a nonnegative integer. Hence, by Theorems 4 and 5, any representation φ has the property that

$$\varphi(x) = \varphi(N_x,e) = (N_x + 1)\varphi(e).$$

Choosing φ so that $\varphi(e)$ is a positive integer establishes the result. QED

Note that, in practice, representations into the rationals, rather than into the integers, are usually used because it is rarely convenient to take as the unit the smallest element of C.

Theorem 8. *Suppose that $\langle A,B,R,o \rangle$ is an extensive system with at least* *one element* a *that is maximal relative to* R *and* o. *Let* A′ *be* A *with all maximal elements deleted,* R′ *be the restriction of* R *to* A′, B′ = {(x,y)| x,yϵA′,(x,y)ϵB,xoyϵA′}, *and* o′ *be the restriction of* o *to* B′. *Then if* A′ *and* B′ *are both non-empty,* < A′,B′,R′,o′ > *is an extensive system with no maximal element.*

Proof. We need only check the axioms of Def. 2, and it is clear that they are all satisfied except, possibly, 4. It could fail if, in the original system, aI(y-x). In that case, however, yIxo(y-x)IxoaIa, which contradicts the choice of yϵA′. QED

Theorem 9. *Suppose that* < A,B,R,o > *is an extensive system with at least one element* a *that is maximal relative to* R *and* o. *Let* < A′,B′,R′,o′ > *be defined as in Theorem 8, suppose that* A′ *and* B′ *are non-empty, and let* C *be any positive real number.*

1. *If there exist* u,vϵA′ *such that* (u,v)ϵB *and* uovRa, *then there exists a positive real-valued function* φ *on* A *such that for all* x,yϵA,

i) xRy *if and only if* $\varphi(x) \geqq \varphi(y)$,
ii) $\varphi(x) = C$ *if* xIa,
iii) *if* (x,y)ϵB′, *then* $\varphi(xoy) = \varphi(x) + \varphi(y)$.

2. *If for all* x,yϵA′, (x,y)ϵB *implies not* xoyRa, *then there exists a positive real-valued function* Φ *on* A *such that for all* x,yϵA

i) xRy *if and only if* $\Phi(x) \geqq \Phi(y)$,
ii) $\Phi(x) = C$ *if* xIa,
iii) *if* (x,y)ϵB,

$$\Phi(xoy) = \frac{\Phi(x) + \Phi(y)}{1 + \Phi(x)\Phi(y)/C^2}.$$

Proof. By Theorems 5 and 8, there exists an additive function φ over A′

1. Choose the unit of φ so that $\varphi(u) + \varphi(v) < C$ and assign $\varphi(x) = C$ if xIa. Parts ii and iii are clearly met. To show i it is sufficient to show that if xϵA′, then $\varphi(x) \leqq \varphi(u) + \varphi(v)$. This is obvious if either uRx or vRx, so we assume that xPu and xPv. By Axiom 4, there exists x-uϵA′ such that (u,x-u)ϵB′ and xIuo(x-u). Thus,

$$\text{uovIaPxIuo(x-u)},$$

from which it follows that vR(x-u) since the contrary leads to a contradiction by Axiom 3. So, by properties of φ,

$$\begin{aligned}\varphi(x) &= \varphi(u) + \varphi(x\text{-}u)\\ &\leqq \varphi(u) + \varphi(v).\end{aligned}$$

2. Define

$$\Phi(x) = \begin{cases} C \tanh \varphi(x) & \text{if aPx} \\ C & \text{if aIx} \end{cases}$$

Since tanh is strictly monotonic increasing and < 1, it is clear that parts i and ii hold. Using elementary properties of tanh, part iii follows for those $x,y \epsilon A'$ for which $(x,y) \epsilon B$ since, by assumption, $(x,y) \epsilon B'$. If $(x,y) \epsilon B$ and either x or yIa, then by Theorem 2, xoyIa, and by substituting $\Phi(a) = C$ in both sides of part iii we see that it still holds. QED

Property 2, iii of Theorem 9 is the well-known formula for the relativistic "addition" of velocities. It is obvious from the proof of part 1 that an additive representation also exists if one is willing to assign ∞ to a maximal element (in the case of velocity, that of light) and extend $+$ in the usual way. Such a representation of velocity fails, however, to have the property that velocity equals the distance traversed divided by the time it takes. Evidently, physicists have preferred to retain the latter derived property and to sacrifice the simple additive representation of concatenation. Perhaps non-additive representations should be kept in mind in other sciences, even when additive ones exist.

In the same vein, it is important to recognize that the representation given in part 2 of Theorem 9 is by no means the only one possible—it is simply the one that has arisen in the theory of relativity. Specifically, let f be any monotonic increasing function that maps the positive reals onto the open interval (0,C) with the property that there exists a function F of two variables such that $f(x+y) = F[f(x),f(y)]$. Then

$$\Phi(x) = \begin{cases} f[\varphi(x)] & \text{if aPx} \\ C & \text{if aIx} \end{cases}$$

has properties 2,i and 2,ii of Theorem 8 and property 2,iii is replaced by:

$$\Phi(xoy) = F[\Phi(x),\Phi(y)].$$

Many results on functional equations of the type $f(x+y) = F[f(x),f(y)]$ are given in Section 2.2 of Aczél (1966).

8. FUNDAMENTAL MEASUREMENT OF TIME DURATION

Although the theory of extensive measurement is widely accepted as a suitable mathematical framework for the fundamental measurement of some basic physical quantities, in particular mass and length, its role in justifying other measures, such as velocity and time, has seemed somewhat less secure to some authors. Of the two, the former has been the less vexing since if both length and time can be measured fundamentally, then velocity can be handled as a derived measure. In fact, as we have seen in Theorem 8, it can also be treated as a fundamental quantity provided that a non-additive representation is accepted.

Time is rarely discussed in detail in connection with presentations of fundamental measurement schemes, and some authors seem to believe that it cannot be handled by extensive measurement methods. In fact, however, Campbell (pp. 550–553 of the 1957 edition) outlined a suitable interpretation for measuring time durations, but this seems to have been overlooked, perhaps because his exposition is a bit opaque. It may, therefore, be worth restating it here. The entities of A are (the periods of) a family of pendulums and their concatenations as defined below. The ordering R is determined as follows: if pendulums x and y are started at exactly the same time (say by placing them side by side and releasing them together by dropping a supporting rod) and if x fails to complete its first period before y completes its first period, then xRy. The concatenation xoy denotes any pendulum with the following property: if xoy and x are released at the same time and if y is released exactly when x completes its first period (doing this would be non-trivial in practice), then xoy and y will complete their first periods at exactly the same time.

With these interpretations, it is no more difficult to be convinced that the axioms of extensive measurement (with no maximal element) are fulfilled than it is with the usual interpretations for mass and length. As there, some care is needed in the choice of pendulums to be sure that Axiom 4 is satisfied. Moreover, our modification of the axioms to permit restrictions on the freedom to concatenate possesses practical advantages—temporal rather than spatial—similar to those for mass and length. The construction of a standard series based upon n concatenations of a duration with itself is especially simple: one merely counts off n periods of the pendulum (of course, one must verify that any two periods of an uninterrupted sequence are equivalent in the sense that each matches the first period of some other pendulum).

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2. The first author worked on the problem both at the University of Pennsylvania and at the Center for Advanced Study in the Behavioral Sciences. During the earlier phase he received partial support from National Science Foundation Grant GB-1462 to the University of Pennsylvania, and during the later phase he was a National Science Foundation Senior Postdoctoral Fellow.

3. The second author was a Fellow of the Miller Institute for Basic Research in Science at the University of California, Berkeley, during the period of this work.

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